Chromatic number, orientations and subtrees

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Abstract. In the eighties, Burr proved that if a graph G has chromatic number least $(t-1)^2$, then every orientation of G contains every oriented tree of order t. He conjectured that the same holds if $(t-1)^2$ is replaced by $2t-2$. We present some evidence towards this conjecture, showing that if G has chromatic number k and order n then every orientation of G contains every oriented tree of order $k/log_2\, n$.

Keywords: Oriented graphs · Chromatic number · Trees · Digraphs.

1 Introduction

The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that the vertex set of G may be partitioned into k parts with no edge of G joining vertices in the same part. It is a folklore result in graph theory that every graph G contains every tree of order $\chi(G)$, and complete graphs contain no larger trees. We are interested in the following Ramsey-type question: if we choose an arbitrary orientation for each edge of G, what (oriented) trees do we expect to find?

If $\vec{T}$ is an oriented graph, we write $G \to \vec{T}$ to indicate that every orientation of G contains an oriented copy of $\vec{T}$. Let P_n denote the *directed path* of order n, i.e., the orientation of an n-vertex path where all edges are directed away from a root vertex. A classical result (independently found four times!) states the following.

Theorem 1. *[4, 5, 7, 8] If G is a graph, then $G \to \vec{P}_{\chi(G)}$.*

We write $\vec{r}(\vec{T})$ for the smallest integer such that $\chi(G) \geq \vec{r}(\vec{T})$ implies $G \to \vec{T}$. The theorem above is thus equivalent to the statement $\vec{r}(\vec{P}_n) \geq n$. In the eighties, Burr established the value of this parameter for oriented stars, and also obtained a general bound which holds for every tree. We write S_t^+ for the oriented star of order t in which each arc is oriented towards a leaf, and S_t^- for the oriented star obtained reversing the arcs of S_t^+.

Theorem 2. *[3] If $t > 2$, then* $\vec{r}(\vec{S}) = \begin{cases} 2t-2 & \text{if } S \text{ is either } S_t^- \text{ or } S_t^+, \\ 2t-3 & \text{otherwise.} \end{cases}$

Theorem 3. *[3] Let G be a graph of chromatic number k, and let $\vec{T}$ be an oriented tree of order t. If $k \geq (t-1)^2$, then $G \to \vec{T}$.*

Abusing notation, let $\vec{r}(t)$ denote the smallest integer k such that $G \to \vec{T}$ for every oriented tree $\vec{T}$ of order t and every graph G with $\chi(G) \geq k$. The theorem above is thus equivalent to the statement $\vec{r}(t) \leq (t-1)^2$. Note that this bound is quadratic in terms of t, whereas the bounds in Theorems 1, 2 and 3 it is linear. Burr also posed the following conjecture.

Conjecture 1. [3] If $t > 2$, then $\vec{r}(t) \leq 2t - 2$.

Burr's conjectured bound is best possible, in the following sense: $K_{2t-3} \not\to S$ if S_t^+ is an oriented star of order t where all arcs are oriented towards leaves. (To see this, consider a regular orientation of K_{2t-3}, i.e., an orientation where each vertex has the same in- and outdegrees.) The bound in Theorem 3 has been improved in the last decade.

Theorem 4. *[1] If $t > 2$ then $\vec{r}(T) \leq \binom{t}{2} + 1$.*

2 Contribution

Our theorem implies that almost every graph G with chromatic number k is such that $G \to T$ for every oriented tree of order $k^{1-o(1)}$. More precisely, the next theorem states that for all positive C and sufficiently large k, if G is a graph with chromatic number k and order n, where $n \leq \mathrm{e}^{(\log k)^C}/2$, then $G \to T$ for every oriented tree T of order $k/(\log k)^C$. (The required inequality holds for almost every graph of order n, as proved by Bollobás [2].)

Theorem 5. *[6] Let G be a graph with order n and chromatic number k. Then $G \to \vec{T}$ for every oriented tree of order $k/\log_2 n$.*

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