On total chromatic number of circulant graphs*

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Abstract. In this work we investigate the total chromatic number of circulant graphs $C_n\left(1,\left\lfloor\frac{n}{2}\right\rfloor\right)$. We present some previous results about the total coloring and prove that the graph $C_7(1,3)$ is Type 2.

Keywords: total coloring · circulant graph · regular graph

1 Introduction and main result

A k-vertex coloring of a graph G is an assignment of k colors to the vertices of G so that adjacent vertices have different colors. The chromatic number $\chi(G)$ is the smallest k for which G has a k-vertex coloring. A k-edge coloring of a graph G is an assignment of colors to the edges so that adjacent edges have different colors. The chromatic index $\chi'(G)$ is the smallest k for which G has a k-edge coloring.

Total colorings combine the vertex and edge colorings by coloring both vertices and edges of a graph G so that adjacent elements (vertices and edges) have different colors. A k-total coloring of a graph G is a total coloring that uses k colors, and the total chromatic number $\chi''(G)$ is the smallest k for which G has a k-total coloring. The Total Coloring Conjecture (TCC) was proposed by Vizing and Behzad [1] and states that for any simple graph G, $\chi''(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G.

The TCC implies that for any simple graph G, $\chi''(G) = \Delta(G)+1$ and in this case we say that G is Type 1; or $\chi''(G) = \Delta(G)+2$ then G is said to be Type 2. The TCC was verified for several classes of graphs.

A circulant graph $C_n(d_1, d_2, \cdots, d_l)$ has vertex set $V = \{1, 2, \cdots, n-1\}$ and edge set $E(G) = \bigcup_{i=0}^{l} E_i(G)$ where $E_i(G) = \{e_0^i, e_0^i, \cdots, e_{n-1}^i\}$ and $e_j^i = (v_j, v_{j+d_i \mod n})$. An edge of $E_i(G)$ is called edge of length d_i. Khennoufa and Togni [3] proved that all members of infinite families of circulant graphs are Type 1.

The main goal of this current work is to determine the total chromatic number of all circulant graphs $C_n\left(1,\left\lfloor\frac{n}{2}\right\rfloor\right)$. It is well known by Chetwynd [2] that when n is even the graph is Type 2. So, we focus on the odd case.

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For $n = 5$, the circulant graph $C_5(1,2)$ is the complete graph K_5 that is Type 1. We prove that $C_7(1,3)$ is Type 2, by using the following theorem to prove that this graph does not have any 5-total coloring.

Theorem 1. *If a k-regular graph G does not have maximal macthing of maximum lenght $\left\lfloor \frac{|E|}{k+1} \right\rfloor$, then does not exists $(k+1)$-total coloring of G [4].*

Theorem 2. *The graph $C_7(1,3)$ is Type 2.*

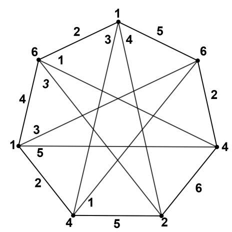

Fig. 1. The graph $C_7(1,3)$ with a 6-total coloring.

Proof. Suppose that $C_7(1,3)$ has a maximal matching M of lenght $\left\lfloor \frac{14}{5} \right\rfloor = 2$. If an edge e in the cycle $v_1 v_2 \cdots v_7 v_1$ belongs to M, then we have a connected component with 7 edges and 5 vertices to match. In any case, we do not have a maximal matching of lenght 2. If an edge $e = v_i v_{i \pm 3 \mod 7} \in M$, then we have a connected component with an edge having 6 adjacent edges, and following that it remains 2 edges to obtain a maximal matching (of lenght 3). Then $C_7(1,3)$ does not have a maximal matching of lenght 2. Since the graph has a total coloring with 6 colors (Figure 1), it is Type 2. $\square$

2 Future Work

The goal of this work is to determine the total chromatic number of infinite families of circulant graphs, contributing to the Total Coloring Conjecture and to the state of the art in this area.

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