Intersection of longest paths in 4-connected graphs

Juan Gutiérrez[1][0000−0002−1860−7383]

[1] Universidad de Ingeniería y Tecnología (UTEC), Perú
[2] jgtierreza@utec.edu.pe

Abstract. It is known that every pair of longest paths in a connected graph intersect each other in at least one vertex. Hippchen [1] conjectured that, for k-connected graphs, every pair of longest paths intersect each other in at least k vertices and prove it for $k = 3$. In this paper we prove Hippchen's conjecture for $k = 4$. We also show, for every $k > 0$, a family of k-connected graphs in which there is a pair of longest paths intersecting each other in exactly k vertices.

Keywords: k-connected graph · longest path · intersection.

1 Main Theorem

All graphs in this paper are simple and notation used is standard [2]. We begin by showing a useful lemma. For space reasons, the proof of it is not presented.

Lemma 1. *Let P and Q be two longest paths in a graph G. Let $u \in V(P) \cap V(Q)$. Let $v \in V(P) \setminus \{u\}$ be such that $P[u, v]$ contains no vertex of $V(Q) \setminus \{u\}$. Let $w \in V(Q) \setminus \{u\}$ be such that $P[u, w]$ contains no vertex of $V(P) \setminus \{u\}$. Then, there is no vw-path internally disjoint from P and Q.*

Theorem 1. *Every pair of longest paths in a 4-connected graph intersect each other in at least four vertices.*

Proof Sketch. Let G be a 4-connected graph and let P and Q be two longest paths in G. Suppose by contradiction that $|V(P) \cap V(Q)| < 4$. As G is 3-connected, P and Q intersect in exactly three vertices [1, Lemma 2.2.3], say a, b and c. Suppose, without loss of generality, that abc is a subsequence in P. Without loss of generality we have two cases, depending on the ordering in which a, b and c appear in Q. Also, as G is 4-connected, the graph $G - \{a, b, c\}$ is connected. Hence, by Lemma 1, and without loss of generality, we have two cases, stated in Fig 1. In each of these cases we obtain a contradiction.

2 Tight Families

As Hippchen mentioned [1, Figure 2.5], in the graph $K_{k,k+2}$, there exists a pair of longest paths intersecting each other in exactly k vertices. As $K_{k,k+2}$ is k-connected, this make the conjecture tight. In this section we show that in fact there is an infinite family of graphs, for every k, that make the conjecture tight.

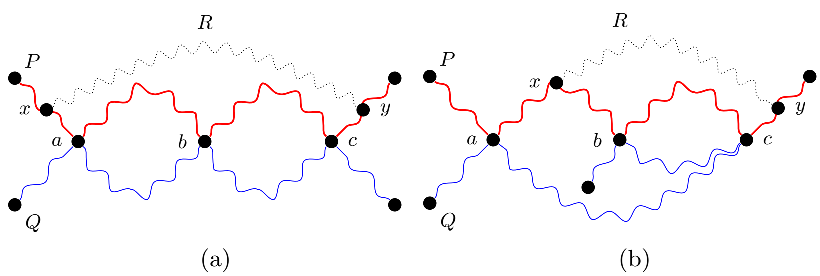

Fig. 1. Cases in the proof of Theorem 1. In both cases, we obtain two paths whose lengths sum $|P|+|Q|+2|R|$, which is a contradiction, as P and Q are longest paths. (a) Paths $P_x \cdot R \cdot P_{yc} \cdot Q_{cb} \cdot Q_{ba} \cdot Q_a$ and $P_y \cdot R \cdot P_{xa} \cdot P_{ab} \cdot P_{bc} \cdot Q_c$, (b) paths $P_a \cdot P_{ax} \cdot R \cdot P_{yc} \cdot P_{cb} \cdot Q_b$ and $Q_a \cdot Q_{ac} \cdot Q_{cb} \cdot P_{bx} \cdot R \cdot P_y$.

Theorem 2. *For every k-connected graph, there exists an infinite family of graphs with a pair of longest paths intersecting each other in exactly k vertices.*

Proof Sketch. Let $S = \{s_1, s_2, \ldots, s_k\}$, and ℓ be a positive integer. For every $i \in [k+1]$, let $X_i = \{a_{i1}, a_{i2}, \ldots, a_{i\ell}\}$ and $Y_i = \{b_{i1}, b_{i2}, \ldots, b_{i\ell}\}$. Let G be a graph with $V(G) = S \cup \{X_i : i \in [k+1]\} \cup \{Y_i : i \in [k+1]\}$, and $E(G) = \{sv : s \in S, v \in V(G) \setminus S\} \cup \{a_{ij}a_{i(j+1)} : i \in [k+1], j \in [\ell-1]\} \cup \{b_{ij}b_{i(j+1)} : i \in [k+1], j \in [\ell-1]\}$ (Fig. 2). It is easy to see that G is k-connected and that $P = a_{11}a_{12}\cdots a_{1\ell}s_1a_{21}a_{22}\cdots a_{2\ell}s_2\cdots a_{k1}a_{k2}\cdots a_{k\ell}s_ka_{(k+1)1}a_{(k+1)2}\cdots a_{(k+1)\ell}$ and $Q = b_{11}b_{12}\cdots b_{1\ell}s_1b_{21}b_{22}\cdots b_{2\ell}s_2\cdots b_{k1}b_{k2}\cdots b_{k\ell}s_kb_{(k+1)1}b_{(k+1)2}\cdots b_{(k+1)\ell}$ are both longest paths, intersecting in exactly k vertices.

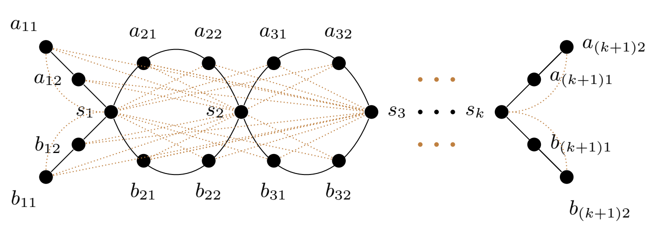

Fig. 2. The graph used in the construction of Theorem 2, in the case $\ell = 2$.

References

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2. Bondy, J. A. and Murty, U. S. R. *Graph Theory*, volume 244 of *Graduate texts in mathematics*. Springer, 2008.