Token Swap on Cographs

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Introduction

Let $G = (V, E)$ be a graph with $|V|$ vertices and $|E|$ edges, with distinct tokens placed on it's vertices. The objective is to reconfigure this initial token placement called $f_0 : V \mapsto V$ into the identity token placement f_i, that maps every node to itself, through a sequence of pairs of adjacent graph vertices that swap the tokens between these vertices. The aim is to know if it is possible to have a swap sequence S that achieve the objective in k or less swaps, with $k \in \mathbb{N}$.

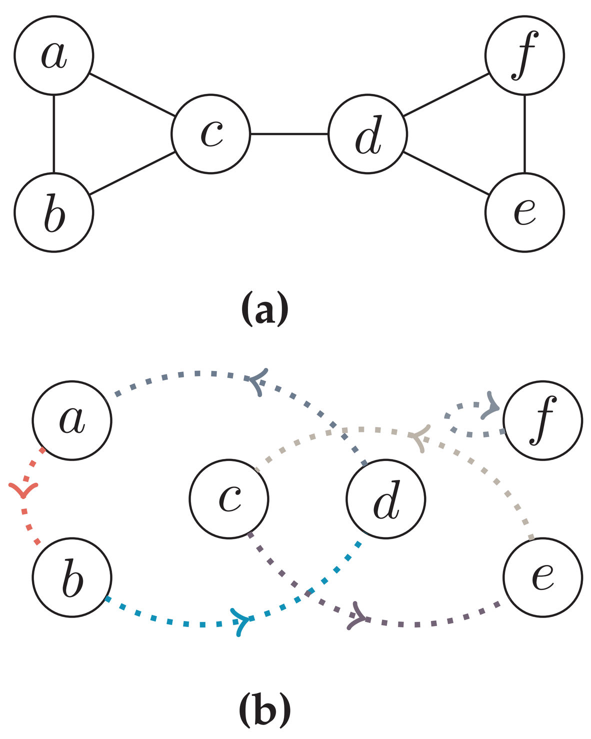

Figure 1: TS instance with graph 1a and token configuration 1b.

The problem was shown to be polynomial time solvable for some graph classes, but only for very special cases [1, 2, 3, 4, 5]. Applications of the TS problem encompass a wide range of fields. From computing efficient interconnection network structures, [6], computational biology [7, 8], modelling Wireless Sensor Networks (WSS) [9], protection routing [10] to qubit allocation for quantum computers [11, 12].

Swapping Tokens on Cographs

The *Cycle Matching Graph* H of a cograph G has each cycle on C^0 as vertice set and two vertices are adjacent if the lowest common ancestor of all vertice pairs in the vertice union in $T(G)$ is an 1-node. Let $\mu(H)$ be the maximum matching in this graph.
It is possible to prove that each independent cycle $C \in CS$ can be solved in $|C| + 1$ or $|C| - 1$ swaps depending on whether this cycle is part of C^0 or C^1, respectively. Also, it is possible to show that cycle interaction is restricted in the best-case scenario and the best improvement on swaps can be calculated on the value of the maximum matching of the cycle matching graph H.
The following theorem implies the polynomial time solvability of `Token Swap` for cographs.

Theorem. *Let G be a cograph with an initial token placement f_0. The minimum number of required swaps is given by $|V(G)| + |C^0| - |C^1| - 2 \times |\mu(H)|$.*

This behavior is also being used to find more efficient algorithms in other graph classes like bipartite chain, wheel and gear. Each possible swap is either called a *merge* or a *split* and changes the cycle set $CS(CG_f)$ by merging two cycles or splitting a cycle into two, respectively. By understanding the interactions of merge and split swaps over the two classes of cycles in the cycle set and initial configuration, it is possible to achieve a polynomial time algorithm for Token Swap in Cographs. Figure 2 shows an example of an swap that merges two cycles of a configuration.

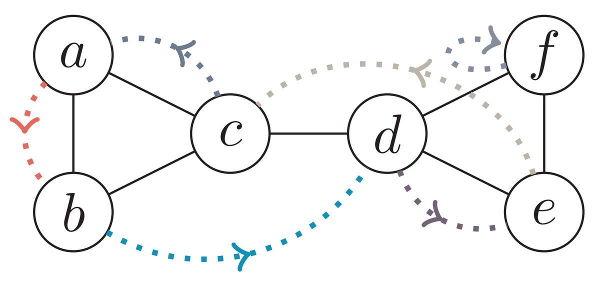

Figure 2: Representation of the instance of Figure 1 after the application of a *merge* swap (c, d).

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Preliminaries

A Conflict Graph $CG_f := (V(G), E_{CG})$ is a digraph that, for a token placement f of a graph G, an edge $(u, v) \in E_{CG}$ if and only if $f(u) = v$. Each node has outdegre 1 and the digraph may contain self-loops.

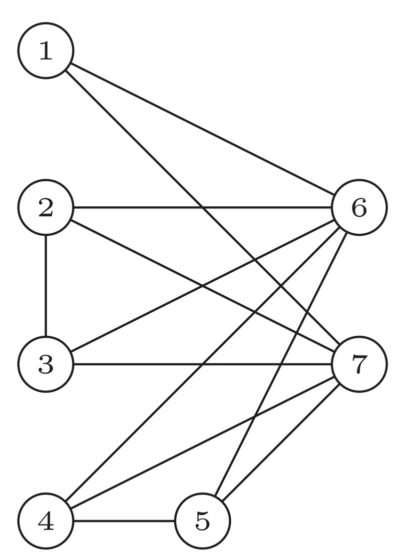

Figure 3: Example of a cograph.

A cograph is defined recursively as follows: a graph on a single vertice is a cograph; if $G_1, G_2, \ldots, G_k$ are cographs, then so is their disjoint union; if G is a cograph, then so is its complement $\overline{G}$. A cotree $T(G)$ of a cograph $G = (V, E)$ is a rooted tree representing it's structure. The leaves of $T(G)$ are exactly V and each internal node is either a 0-node and 1-node. The children of an 1-node are 0-nodes or leaves and the children of a 0-node are 1-nodes or leaves. Two vertices are adjacent in a cograph if and only if their lowest common ancestor is an 1-node.
The set of permutation cycles of CG for f is defined as $CS(CG_f) = \{C_1, C_2, \ldots, C_k\}$. Let $C^1 \subseteq CS$ be the set of cycles that have a lowest common ancestor of all vertice pairs of $V(C)$ as an 1-node in the cotree or is a cycle of size one and let $C^0 = CS \setminus C^1$.

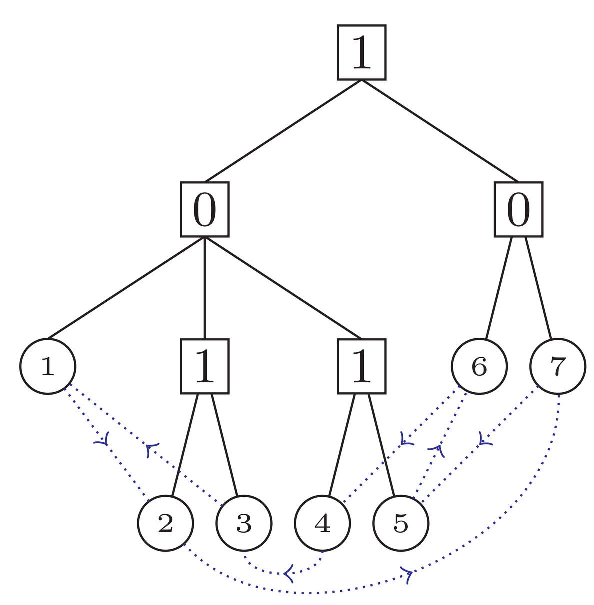

Figure 4: Cotree and conflict graph joint representation.

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