Valid inequalities for the Integer Knapsack Cover polyhedron with setup constraints*

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Introduction

The Integer Knapsack Cover set is defined as $X = \{x \in Z^n_+ : \sum_{i \in N} a_i x_i \geq b\}$ for some positive integer n, where $N = \{1, \ldots, n\}$. We assume that $a_1, \ldots, a_n, b$ are positive integers satisfying $a_1 < a_2 < \ldots < a_n < b$ (see [4]). The inequality defining X is called *cover constraint*.

Let m and M be fixed parameters such that $m < b < M$. We are interested in the convex hulls of the sets (1) and (2). These sets appear as a substructure of the cutting stock problem (CSP) polyhedron if the demand b can be satisfied with overproduction and setup constraints are considered (X_S) and minimum lot sizes are also imposed (X_{SM}) [3]. In this context, y_i is a binary variable indicating if cutting pattern i is used or not, x_i indicates the number of times a cutting pattern is used, and a_i is the number of itens in the cutting pattern i.

$$X_S = \{(x, y) : x \in X, y \in \{0,1\}^n, x_i \leq My_i, \forall i \in N\} \tag{1}$$

$$X_{SM} = \{(x, y) : x \in X, y \in \{0,1\}^n, my_i \leq x_i \leq My_i, \forall i \in N\} \tag{2}$$

Throughout this work, e_i denotes the ith canonical vector of $\mathbb{R}^n$ and depending on the context, $\mathbb{1}$ can denote the n-dimensional all-ones vector.

On the dimension of conv(X_S) and conv(X_{SM})

- Mazur [1] first proved that conv(X) is full-dimensional. One way to prove the full-dimensionality of conv(X) is to take an integer feasible set $Y = \{z_1, y_1, y_2, \ldots, y_n\}$ of distinct points in which z_1 belongs to the x_1-axis and each y_i belongs to the x_i-axis. Such a set can be proven to be affinely independent. We extend Mazur's result for both conv(X_S) and conv(X_{SM}).

Proposition 1 *For sufficiently large M, conv(X_{SM}) is full-dimensional.*

Proof. Consider the following $2n + 1$ feasible integer points.

$$\begin{aligned} w^i &= (\alpha_i e_i, e_i), & \alpha_i &= m\left\lceil \frac{b}{a_i} \right\rceil + 1, & \forall i \in N \\ z^i &= (\beta_i e_i, e_i), & \beta_i &= m\left\lceil \frac{b}{a_i} \right\rceil + 2, & \forall i \in N \\ z &= \sum_{i \in N} (\lambda_i e_i, e_i), & \lambda_i &= m\left\lceil \frac{b}{a_i} \right\rceil, & \forall i \in N. \end{aligned}$$

- Taking the linear combination (3) leads to the pair of linear equations (4).

$$\sum_{i=1}^{n} \gamma_i w_i + \sum_{i=1}^{n} \delta_i z_i + \delta z = 0, \text{ where } \sum_{i=1}^{n} (\gamma_i + \delta_i) + \delta = 0. \tag{3}$$

$$\begin{cases} \sum_{i=1}^n (\gamma_i \alpha_i + \delta_i \beta_i) e_i + \sum_{i=1}^n \delta \lambda_i e_i = 0 \\ \sum_{i=1}^n (\gamma_i + \delta_i) e_i + \sum_{i=1}^n \delta e_i = 0. \end{cases} \tag{4}$$

- A system of equations (5) on the coefficients $\gamma_i, \delta_i, \delta$ can be obtained by the linear independence of $\{e_i\}_i$, and for which the only solution is $\gamma_i = \delta_j = \delta = 0$, for all $i, j \in N$.

$$\begin{cases} \sum_{i=1}^n (\gamma_i \alpha_i + \delta_i \beta_i + \delta \lambda_i) e_i = 0 \\ \sum_{i=1}^n (\gamma_i + \delta_i + \delta) e_i = 0 \end{cases} \tag{5}$$

- Hence, if follows that the set of points $\{w^i : i \in N\} \cup \{z_i : i \in N\} \cup \{z\}$ is affine independent.

□

Corollary 1 *For sufficiently large M, conv(X_S) is full-dimensional*

Proof. It follows from the fact that $X_{SM} \subset X_S$.

□

On the facets of conv(X_S) and conv(X_{SM})

- For the polyhedron conv(X_S), we introduce a class of valid inequalities. We claim that for each $j \in N$, the inequality (6) is valid for conv(X_S).

$$\sum_{i \in N \setminus \{j\}} y_i + \frac{1}{\lceil b/a_j \rceil} x_j \geq 1 \tag{6}$$

- Indeed, by the assumption $b > 0$, at least one item must be used so the demand can be satisfied. Fix one item j. If some other item $i \in N \setminus \{j\}$ is used, then $y_i = 1$ and inequality (6) holds. If none of the items $i \in N \setminus \{j\}$ are used, the only item left is j and we need at least $\lceil b/a_j \rceil$ copies of j to satisfy the demand. In this case, (6) also holds.

Theorem 1 *For each $j \in N$, the valid inequality (6) is facet-defining for conv(X_S).*

Proof.

- Let $\alpha_i := \lceil b/a_i \rceil$, for all $i \in N$ and for some $k \neq j$, consider the following points of X_S.

$$\begin{aligned} w^i &= (\alpha_i e_i, e_i), \quad i = 1, \ldots, n \\ z^i &= (2\alpha_i e_i, e_i), \quad i = 1, \ldots, n, \quad i \neq j \\ z &= (\alpha_k e_k, e_k + e_j) \end{aligned} \tag{7}$$

- All of the points in (7) satisfy (6) at equality, since for each of them, only one item is used. We will show that they are are affinely independent by proving that they are linearly independent. Let (8) be an arbitrary linear combination.

$$\sum_{i \in N} \beta_i w^i + \sum_{i \in N \setminus \{j\}} \gamma_i z^i + \delta z = 0 \tag{8}$$

- Rewriting (8) in terms of e_i's and linear independence of $\{e_i\}_i$ lead to (9), a system that only admits the trivial solution.

$$\begin{aligned} \beta_j &= 0 \\ \beta_k + 2\gamma_k + \delta &= 0 \\ \beta_i + 2\gamma_i &= 0, \quad i \in N \setminus \{j, k\} \\ \beta_j + \delta &= 0 \\ \beta_k + \gamma_k + \delta &= 0 \\ \beta_i + \gamma_i &= 0, \quad i \in N \setminus \{j, k\} \end{aligned} \tag{9}$$

□

- For the polyhedron conv(X_{SM}), we prove that the minimum lot sizes constraint is facet-defining.

Theorem 2 *For each $j \in N$, the inequality $x_j \geq my_j$ is facet-defining for conv(X_{SM}).*

Proof.

- Let $\alpha_i := \lceil b/a_i \rceil$, for all $i \in N$ and consider the following points of X_{SM} satisfying $x_j = my_j$.

$$w^i = (\alpha_i e_i, e_i), \quad \forall i \neq j \tag{10}$$

$$z^i = (2\alpha_i e_i, e_i), \quad \forall i \neq j \tag{11}$$

$$w = \left(me_j + \sum_{i \in N \setminus \{j\}} \alpha_i e_i, \mathbb{1}\right) \tag{12}$$

$$z = \left(\sum_{i \in N \setminus \{j\}} \alpha_i e_i, \mathbb{1} - e_j\right) \tag{13}$$

- Let $(\mu, \lambda)(x, y) = \mu_0$ be a generic hyperplane containing these points. We will show that this hyperplane must be $x_i - my_i = 0$.
- By applying the points (10), (11), (12) e (13) into the hyperplane equation, we obtain the relations (14), (15), (16) and (17).

$$\alpha_i \mu_i + \lambda_i = \mu_0, \quad \forall i \neq j \tag{14}$$

$$2\alpha_i \mu_i + \lambda_i = \mu_0, \quad \forall i \neq j \tag{15}$$

$$m\mu_j + \sum_{i \in N \setminus \{j\}} \alpha_i \mu_i + \sum_{i \in N} \lambda_i = \mu_0 \tag{16}$$

$$\sum_{i \in N \setminus \{j\}} \alpha_i \mu_i + \sum_{i \in N \setminus \{j\}} \lambda_j = \mu_0 \tag{17}$$

- Equations (14), (15) and (17) imply $\mu_i = \lambda_i = 0$, for all $i \neq j$ and $\mu_0 = 0$. By setting to zero these coefficients in equation (16), we conclude that $m\mu_j + \lambda_j = 0$, i. e. the initial hyperplane equation must be $\mu_j x_j - (m\mu_j) y_j = 0$.

□

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