Intersection of longest paths in 4-connected graphs

Juan Gutiérrez†

†Department of Computer Science, Universidad de Ingeniería y Tecnología, Perú

Hippchen's Conjecture

Known result: Every pair of longest paths in a **connected** graph intersect in at least **one vertex**.

Conjecture 1 ([2, Conjecture 2.2.4]). *Every pair of longest paths in a **k-connected** graph intersect in at least **k vertices**.*

- Proved by himself for $k = 3$
- Our result: extended for $k = 4$
- A similar conjecture, for cycles instead of paths, was proposed by Grötschel and attributed to Scott Smith [1, Conjecture 5.2].

Auxiliar lemma

Lemma 2. *Let P and Q be two longest paths in a graph G. Let $u \in V(P) \cap V(Q)$. Let v be a vertex in $V(P) \setminus V(Q)$ such that $P[u,v]$ is internally disjoint from Q. Let w be a vertex in $V(Q) \setminus V(P)$ such that $Q[u,w]$ is internally disjoint from P. Then, there is no vw-path internally disjoint from both P and Q.*

Main theorem

Theorem 3. *Every pair of longest paths in a 4-connected graph intersect in at least four vertices.*

Proof sketch.

- Let P and Q two longest paths.
- Let $\{a,b,c\}$ be the intersection of P and Q; suppose abc is a subsequence in P
- We divide the proof in two cases, where $\{a,b,c\}$
  Case 1. abc is a subsequence in Q.
  Case 2. acb is a subsequence in Q.
- In each case we define
  $G' = G - \{a,b,c\}$
  $E(H) = \{XY :$
  there is a X-Y path in G', with no internal vertex in $V(P \cup Q)\}$.
- By the Auxiliar Lemma, some edges cannot exist in H, which lead us to a contradiction.

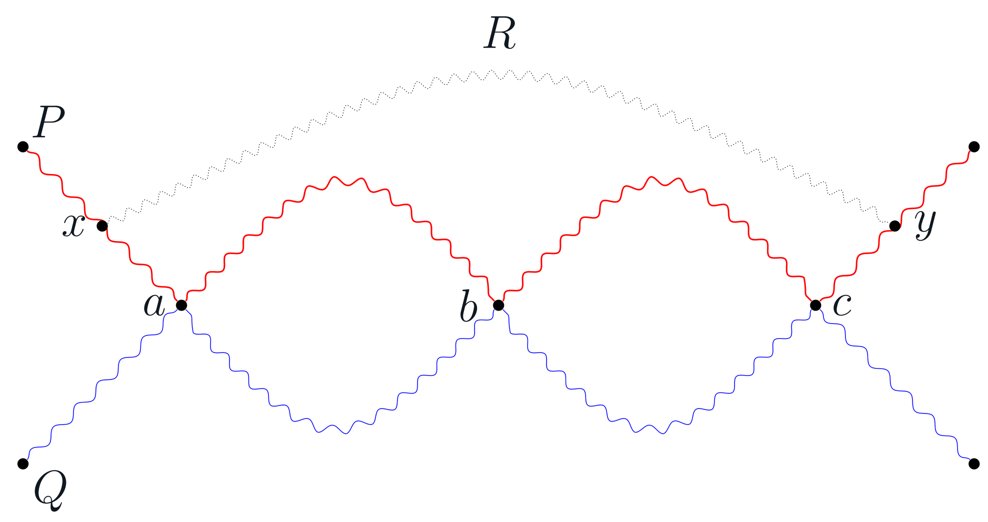

Fig. 1: abc is a subsequence in Q.

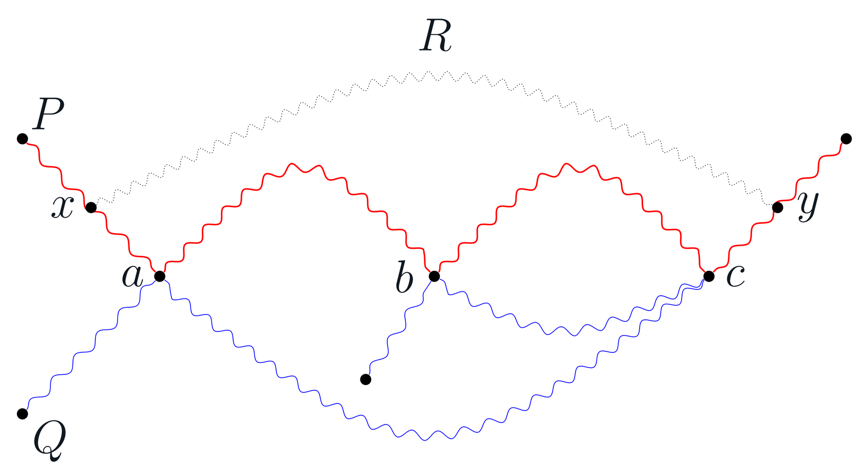

Fig. 2: acb is a subsequence in Q.

Tight families

Hippchen's Conjecture is tight for any k: $K_{k,2k+2}$. We generalized this result.

Theorem 4. *For every k-connected graph, there exists an infinite family of graphs with a pair of longest paths intersecting each other in exactly k vertices.*

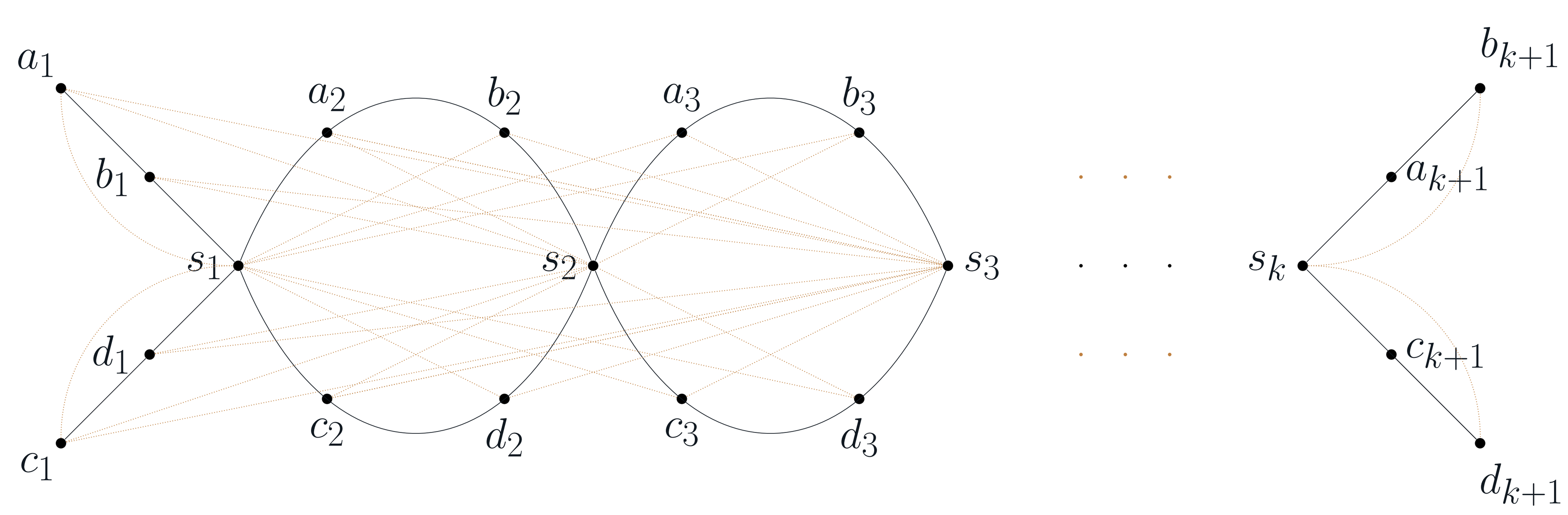

Fig. 3: The graph used in the construction of Theorem 4 when $\ell = 2$.

References

[1] M. Grötschel. "On intersections of longest cycles". In: *Graph Theory and Combinatorics* (1984), pp. 171–189.

[2] T. Hippchen. "Intersections of Longest Paths and Cycles". MA thesis. Georgia State University, 2008.