Linial's Conjecture for Matching-Spine Digraphs

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Definitions

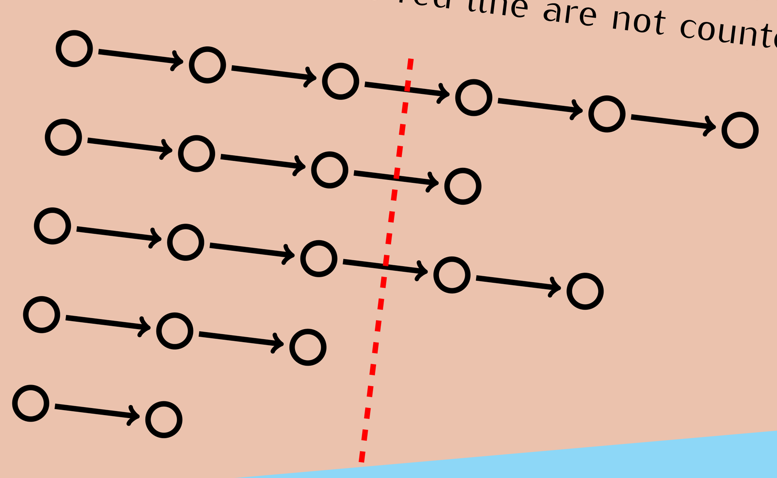
For a digraph D, let $V(D)$ denote its vertex set and let $A(D)$ denote its arc set. Let P be a path. We denote by $V(P)$ the set of vertices of P. The order of P, denoted by $|P|$, is the number of its vertices.

A *path partition* $\mathcal{P}$ of a digraph D is a set of disjoint paths which cover $V(D)$. Let $\pi(D)$ denote the cardinality of a minimum path partition of D. Given a positive integer k, the *k-norm* of $\mathcal{P}$ is defined as $\sum_{P\in\mathcal{P}} \min\{|P|, k\}$. A path partition of minimum k-norm is called *k-optimal* and its k-norm is denoted by $\pi_k(D)$. Note that $\pi(D) = \pi_1(D)$.

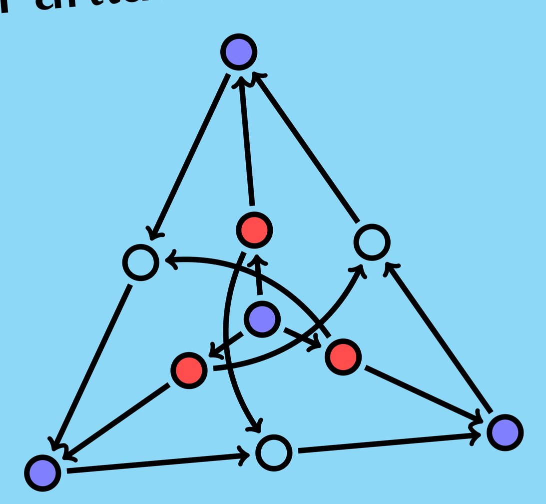
A *stable set* S in a digraph D is a subset of vertices of $V(D)$ such that no two vertices of S are adjacent. Let $\alpha(D)$ denote the cardinality of a maximum stable of D. Let k be a positive integer. A *partial k-coloring* $\mathcal{C}$ of D is a set of k disjoint stable sets. The *weight* of $\mathcal{C}$ is defined as $\sum_{C\in\mathcal{C}} |C|$. A partial k-coloring of maximum weight is called *optimal* and its weight is denoted by $\alpha_k(D)$. Note that $\alpha(D) = \alpha_1(D)$.

k-Norm

In this example, the 3-norm of this path partition is $14 = 3+3+3+3+2$. Note that the vertices on the right side of the red line are not counted in the 3-norm.



Partial k-Coloring



A partial 2-coloring with colors red and blue that has weight $7 = 4+3$.

Linial's Conjecture

In 1950, Dilworth proved that the equality $\pi(D) = \alpha(D)$ holds when D is a transitive acyclic digraph. In 1960, Gallai and Milgram generalized Dilworth's Theorem to arbitrary digraphs establishing that $\pi(D) \le \alpha(D)$ for every digraph D. Later, in 1976, Greene and Kleitman generalized Dilworth's theorem, showing that for every transitive acyclic digraph D and every positive integer k, equality $\pi_k(D) = \alpha_k(D)$ holds. In 1981, Linial conjectured that one could generalize Greene-Kleitman's Theorem to arbitrary digraphs as follows.

Conjecture (Linial,1981)

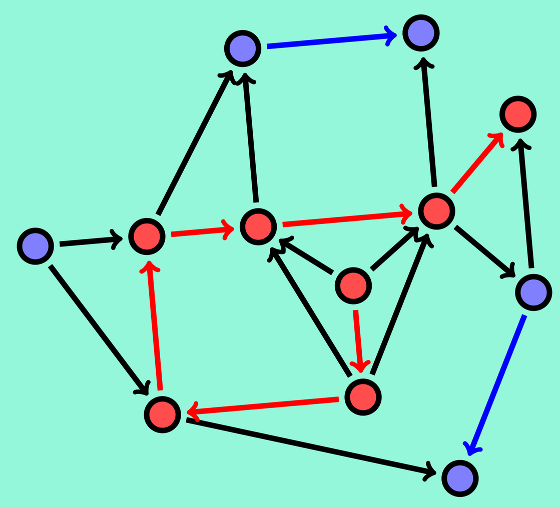
Let D be a digraph and let k be a positive integer. Then $\pi_k(D) \le \alpha_k(D)$.

This conjecture remains open, but there are some particular cases which were already solved.

Matching-Spine Digraphs

Let D be a digraph and let X, Y be a partition of $V(D)$. We say that $D[X, Y]$ is a *matching-spine* digraph if $D[X]$ has a Hamilton path and the arc set of $D[Y]$ is a matching. In this work we give partial results on the validity of Linial's Conjecture for matching-spine digraphs.

Matching-Spine Digraph

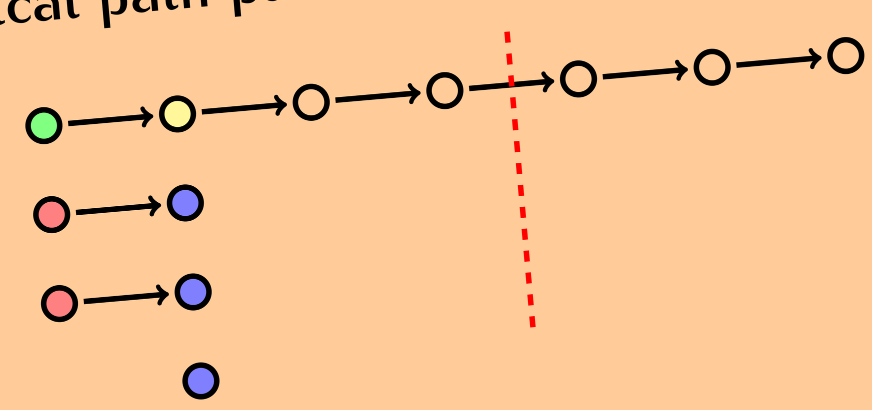


Example of a matching-spine digraph. Red and blue vertices represent sets X and Y, respectively. The red arcs correspond to a Hamilton path of $D[X]$ and the arc set of $D[Y]$, (blue arcs) is a matching.

Canonical Structures

We define a *canonical path partition* $\mathcal{P}$ of D as the one containing a Hamilton path of $D[X]$ together with all maximal paths of $D[Y]$; clearly $\pi_k(D) \le |Y| + \min\{|X|, k\}$. Consider the partition of Y into sets Y^0, Y^+, Y^- such that Y^0 contains the isolated vertices, Y^+ the sources and Y^- the sinks in $D[Y]$. We define a *canonical partial k-coloring* $\mathcal{C}$ of D as the one containing the sets $Y^0 \cup Y^-$, Y^+ and $\min\{|X|, k-2\}$ singletons of X; clearly $\alpha_k(D) \ge |Y| + \min\{|X|, k-2\}$. Therefore $\pi_k(D) \le \alpha_k(D) + 2$.

Canonical path partition and partial k-coloring



A canonical path partition and a canonical partial 4-coloring for the matching-spine digraph in the previous figure. The 4-norm for this path partition is 9. This partial 4-coloring has weight 7.

Our Results

We split the class of matching-spine digraphs into two classes: k-loose and k-tight digraphs. The former class is defined so as to guarantee that $\alpha_k(D) \ge |Y| + \min\{|X|, k\}$, thus ensuring that Linial's Conjecture holds trivially in this case. Our main contribution is a proof that if D is in the latter class, then $\pi_k(D) \le |Y| + \min\{|X|, k-1\}$. The technique used in this proof is a (non-trivial) extension of the one used by Sambinelli, Nunes da Silva and Lee to prove Linial's Conjecture for spine digraphs (a subclass of matching-spine digraphs). Specifically, the technique involves finding a pair of paths whose union contains $|X| + k + 1$ vertices, thus providing a path partition of k-norm $|Y| + \min\{|X|, k-1\}$, one unit smaller than that of a canonical path partition. So, when $\alpha_k(D) \ge |Y| + \min\{|X|, k-1\}$, we guarantee that Linial's Conjecture also holds. The remaining case occurs when $\alpha_k(D) = |Y| + \min\{|X|, k-2\}$. Note that in this case it must be shown that $\pi_k(D) \le |Y| + \min\{|X|, k-2\}$. We believe it is possible to prove the validity of this inequality, which would settle Linial's conjecture for matching- spine digraphs.

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