Coefficients of the solid angle and Ehrhart quasi-polynomials

Fabrício Caluza Machado and Sinai Robins
Instituto de Matemática e Estatística - Universidade de São Paulo
{fabcml, sinai.robins}@gmail.com

arXiv:1912.08017

Abstract

Macdonald developed a discrete volume measure for any rational polytope P, called solid angle sum, which gives a natural discrete volume for P. We give a local formula for the codimension two quasi-coefficient of the solid angle sum of P. We also show how to recover the classical Ehrhart quasi-polynomial from the solid angle sum and in particular we find a similar local formula for the codimension one and codimension two quasi-coefficients. These local formulas are naturally valid for all positive real dilates of P.

An interesting open question is to determine necessary and sufficient conditions on a polytope P for which the discrete volume of P given by the solid angle sum equals its continuous volume: $A_P(t) = \mathrm{vol}(P)t^d$. We prove that a sufficient condition is that P tiles $\mathbb{R}^d$ by translations, together with the Hyperoctahedral group.

Introduction

Given a rational polytope $P \subseteq \mathbb{R}^d$, we consider $L_P(t) := |tP \cap \mathbb{Z}^d|$, the number of integer points in the positive real dilates $tP := \{tx : x \in P\}$, and also the *solid angle sum* $A_P(t) := \sum_{x\in\mathbb{Z}^d} \omega_{tP}(x)$, which counts the integer points weighted by the proportion of the space around that point which the polytope occupies. Ehrhart and Macdonald showed that these quantities can be written as *quasi-polynomials* functions of t, that is, as expressions of the form

$$L_P(t) := |tP \cap \mathbb{Z}^d| = \mathrm{vol}(P)t^d + e_{d-1}(t)t^{d-1} + \cdots + e_0(t),$$

$$A_P(t) := \sum_{x\in\mathbb{Z}^d} \omega_{tP}(x) = \mathrm{vol}(P)t^d + a_{d-1}(t)t^{d-1} + \cdots + a_0(t),$$

where each *quasi-coefficient* $e_k(t)$ and $a_k(t)$ is a periodic function with period dividing the *denominator* of P, defined to be the smallest integer m such that mP in an integer polytope. In particular, if all vertices of P have integer coordinates, these expressions are polynomials for integer values of t.

One of the motivations for the study of these coefficients is that they capture geometric information about the polytope. McMullen [3] proved the existence of functions μ, not unique, such that for a rational P,

$$e_k(t) = \sum_{F\subseteq P,\ \dim(F)=k} \mathrm{vol}^*(F)\mu(tP, tF),$$

where the sum is taken over all k-dimensional faces of P, $\mathrm{vol}^*(F)$ is the k-dimensional volume of the face F normalized by the volume of the fundamental domain of the integer lattice in the linear space parallel to F and μ depends only on "local" geometric data associated to the face F and the polytope P. Such formulas are called *local formulas* for the quasi-coefficients.

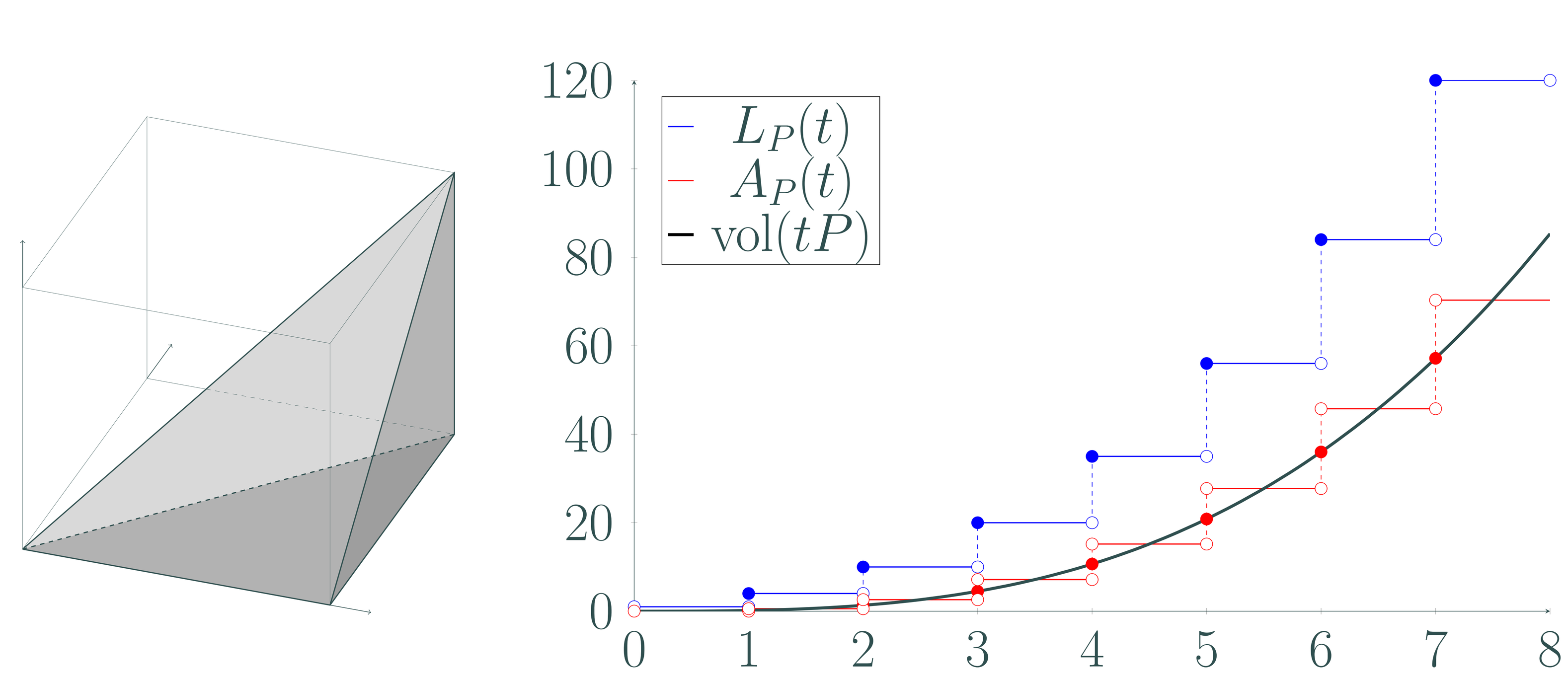

Figure 1: A simplex $P := \mathrm{conv}\{(0,0,0)^\intercal, (1,0,0)^\intercal, (1,1,0)^\intercal, (1,1,1)^\intercal\}$ that tiles the cube together with reflections. Note that $A_P(t) = \mathrm{vol}(tP)$ for integer values of t.

Methods

Denoting the Gaussian function in $\mathbb{R}^d$ as $\phi_{d,\epsilon}(x) := \epsilon^{-d/2}e^{-\pi\|x\|^2/\epsilon}$, the solid angle can be computed with the convolution $\omega_P(n) = \lim_{\epsilon\to 0^+}(\mathbf{1}_P * \phi_{d,\epsilon})(n)$. The method from Diaz, Le, and Robins [2] consists in writing the solid angle sum with these convolutions, applying Poisson summation and then using Stokes formula to reduce the integrals over P to integrals over its facets:

$$\begin{aligned} A_P(t) &= \sum_{n\in\mathbb{Z}^d} \lim_{\epsilon\to 0^+} (\mathbf{1}_{tP} * \phi_{d,\epsilon})(n) = \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d} \hat{\mathbf{1}}_{tP}(\xi)\hat{\phi}_{d,\epsilon}(\xi) \\ &= t^d \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d} \hat{\mathbf{1}}_P(t\xi)\hat{\phi}_{d,\epsilon}(\xi) = t^d \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d} \hat{\phi}_{d,\epsilon}(\xi) \int_P e^{-2\pi i\langle t\xi, x\rangle}\,\mathrm{d}x \\ &= t^d\mathrm{vol}(P) + t^{d-1} \lim_{\epsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} \hat{\phi}_{d,\epsilon}(\xi) \sum_{\substack{F\subseteq P \\ \dim(F)=d-1}} \frac{\langle \xi, N_P(F)\rangle}{-2\pi i\|\xi\|^2} \int_F e^{-2\pi i\langle t\xi, x\rangle}\,\mathrm{d}x, \end{aligned}$$

where $N_P(F)$ is the unit normal vector pointing outward to F. This last step can be repeated until we get a face orthogonal to ξ, when the integral evaluates to the volume of the face. The resulting series can then be solved to produce local formulas for the first few quasi-coefficients $a_k(t)$.

Denoting the quasi-coefficients of a polytope P by $e_k(P;t)$ and $a_k(P;t)$, we prove that

$$e_k(P;t) = \lim_{\tau\to 0^+} a_k\big(P + \tau(P-a); t\big),$$

where a is any point in the interior of P. In this way, the local formulas found for $a_k(t)$ can be converted to local formulas for the quasi-coefficients $e_k(t)$ as well.

Main Results

Our methods offer a further development for the initial approach of Diaz, Le, and Robins [2]. Our main result is an explicit, local formula for the codimension two quasi-coefficient $a_{d-2}(t)$ of the solid angle sum $A_P(t)$ of a rational polytope P. This formula simplifies when we only consider integer polytopes and integer dilations and, in particular, it gives an explicit formula for the solid angle sum of an integer polytope in dimensions 3 and 4, extending Pick's formula valid for dimension 2.

Theorem 1. *Let $P \subseteq \mathbb{R}^d$ be a full-dimensional rational polytope. Then for positive real values of t, the codimension two quasi-coefficient of the solid angle sum $A_P(t)$ has the following finite form:*

$$\begin{aligned} a_{d-2}(t) = \sum_{\substack{G\subseteq P, \\ \dim G=d-2}} \mathrm{vol}^*(G) \Bigg[&\frac{c_G}{2k}\left(\frac{\|v_{F_2}\|}{\|v_{F_1}\|}\overline{B}_2\big(\langle v_{F_1}, \bar{x}_G\rangle t\big) + \frac{\|v_{F_1}\|}{\|v_{F_2}\|}\overline{B}_2\big(\langle v_{F_2}, \bar{x}_G\rangle t\big) \right) \\ &+ \left(\omega_P(G) - \frac{1}{4}\right)\mathbf{1}_{\Lambda_G^*}(t\bar{x}_G) - s\big(h,k;(x_1+hx_2)t, -kx_2t\big)\Bigg]. \end{aligned}$$

In particular, for $d=3$ or 4, if P is an integer polytope in $\mathbb{R}^d$, then for positive integer values of t the solid angle sum is:

$$\begin{aligned} A_P(t) &= \mathrm{vol}(P)t^d \\ &+ \sum_{\substack{G\subseteq P, \\ \dim G=d-2}} \mathrm{vol}^*(G)\left[\frac{c_G}{12k}\left(\frac{\|v_{F_1}\|}{\|v_{F_2}\|} + \frac{\|v_{F_2}\|}{\|v_{F_1}\|} \right) + \omega_P(G) - \frac{1}{4} - s(h,k) \right] t^{d-2}. \end{aligned}$$

Similar formulas are also obtained for the Ehrhart quasi-coefficients $e_{d-2}(t)$ and $e_{d-1}(t)$. In the formulas above, $\overline{B}_2$ is the periodized Bernoulli polynomials of degree 2 and s is a Dedekind-Rademacher sum, which simplifies to a Dedekind sum in the integer case. The other parameters are the "local" information associated to the position of the affine span of a face and the polytope P.

Main references

[1] Nicole Berline and Michèle Vergne. Local Euler-Maclaurin formula for polytopes. *Mosc. Math. J.*, 7(3):355–386, 573, 2007.

[2] Ricardo Diaz, Quang-Nhat Le, and Sinai Robins. Fourier transforms of polytopes, solid angle sums, and discrete volume. *arXiv preprint*, 2016. arXiv:1602.08593.

[3] Peter McMullen. Lattice invariant valuations on rational polytopes. *Arch. Math. (Basel)*, 31(5):509–516, 1978/79.

Funding

F.C.M was supported by grant #2017/25237-4, from the São Paulo Research Foundation (FAPESP). This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico — CNPq (Proc. 423833/2018-9) and by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior — Brasil (CAPES) — Finance Code 001.